Section 6: Two-way and Multi-way ANOVA

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Looking at multiple factors simultaneously allows us to:

1. Study the factors in one experiment instead of multiple experiments
2. Study how conditions interact

- Two factors are crossed if all possible combinations of the factors' levels occur in the design. In this case, we say the design has factorial structure and is a factorial design.
- Interaction: Factors A and B interact if the effect of factor A on the response changes for different values of factor B.

$\boxed{\text{EX}}$ Fabric wear experiment (wear.csv on the webpage)

BF[2]

		Proportion of Filler		
		25%(1)	50%(2)	75%(3)
Filler type	1	155	198	235
		173	177	229
	2	137	129	155
		160	98	132

Model

$$y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \varepsilon_{ijk}$$

i^{th} filler $\quad$ j^{th} prop. $\quad$ k^{th} rep. $(k = 1, 2)$ $\quad$ i^{th} level of filler $i = 1, 2$ $\quad$ j^{th} level of prop. $j = 1, 2, 3$ $\quad$ $(ij)^{th}$ level of interaction

Three Null Hypotheses

H_0: The mean wear for the 2 filler types is the same $\leftarrow$ check main effect for filler

–OR–

H_0: $\alpha_1 = \alpha_2 = 0$ $\quad$ "Basic Factors"

H_0: The mean wear for the 3 proportions is the same $\leftarrow$ check main effect for filler

–OR–

H_0: $\beta_1 = \beta_2 = \beta_3 = 0$

H_0: Proportion and Filler do NOT interact $\leftarrow$ check interaction effect

–OR– $\quad$ "compound factor"

H_0: $(\alpha\beta)_{11} = (\alpha\beta)_{12} = (\alpha\beta)_{13} = (\alpha\beta)_{21} = (\alpha\beta)_{22} = (\alpha\beta)_{23} = 0$

Means for Factor Levels and Cells

Proportion of filler

Filler type	25% (1)	50% (2)	75% (3)	
1	164	187.5	232	194.5
2	148.5	113.5	143.5	135.167
	156.25	150.5	187.75	

$\bar{y}_{13\cdot}$

$\bar{y}_{1\cdot\cdot}$

$\bar{y}_{2\cdot\cdot}$

"Cell (2,1)"

$\bar{y}_{21\cdot}$

$\bar{y}_{\cdot 1\cdot}$ $\bar{y}_{\cdot 2\cdot}$ $\bar{y}_{\cdot 3\cdot}$

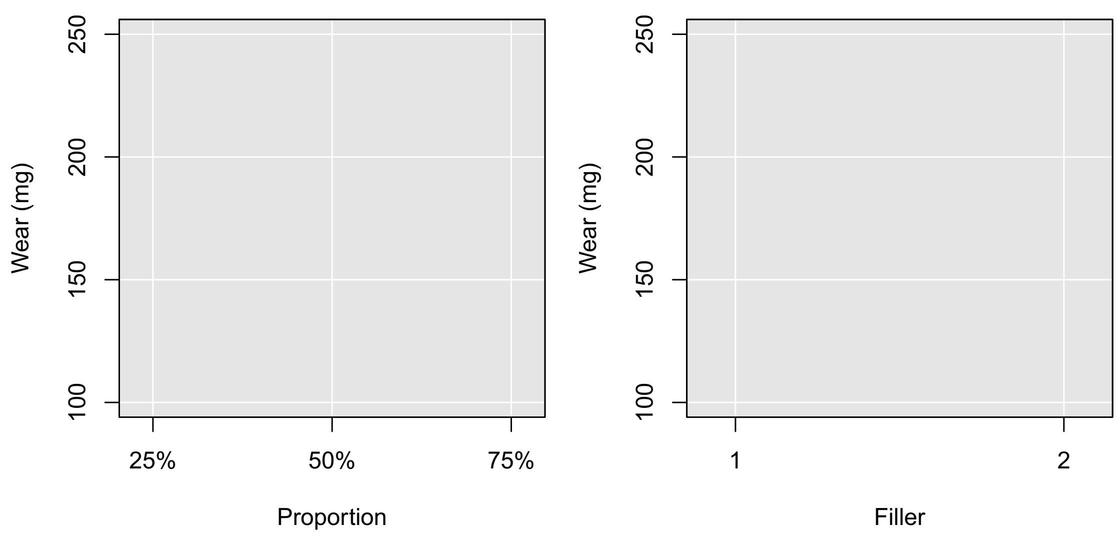
$\bar{y}_{\cdot\cdot\cdot} = 164.833$

Assessing the interaction requires: a response and two crossed factors

- Use an interaction plot

Interaction Plot

Two ways to construct (you build each)



- Which plot do you prefer? Why?
- What do you conclude from the interaction plot?

BF[2] in Experimental Setting
(aka Two-Way Completely Randomized Design CR[2])

- Form treatments from crossing two basic factors
- Treatment combinations are assigned to units completely at random
- For balance, try to use the same number of units for each treatment
- If we want to test the interaction, we should use more than 1 observation per cell

Decomposition of d.f. for BF[2]

1. Decompose: Observed values = Cell means + Residuals
2. Decompose the cell means
   a. Calculate means for the two main effects (aka "basic factors") using the cell means
   Then,
   estimated main effect = factor level mean - grand mean
   b. Estimated interaction effect
   = cell mean - (grand mean + factor A effect + factor B effect)

Note: These rules conform to our general decomp. rule which says to use the average for a factor level minus the sum of effects for all outside factors.

Degrees of freedom in BF[2]:

(suppose factor A has levels $i = 1, \ldots, I$,
and factor B has levels $j = 1, \ldots, J$)

Source	df
Grand Mean	1
Basic factor A	$df_A = I - 1$
Basic factor B	$df_B = J - 1$
A × B Interaction	$df_A \times df_B = (I-1)(J-1)$
Residual	$\underbrace{IJ}_{\text{Total \# of cells}} \times (\underbrace{n}_{\text{\# of obs. per cell}} - 1)$

Note: df_{Residual} is also written "N - IJ"

total # of units

Means for Factor Levels and Cells

Proportion of filler

Filler type	25% (1)	50% (2)	75% (3)	
1	164	187.5	232	194.5
2	148.5	113.5	143.5	135.167
	156.25	150.5	187.75	

$\bar{y}_{13\cdot}$ $\bar{y}_{1\cdot\cdot}$ $\bar{y}_{2\cdot\cdot}$ $\bar{y}_{21\cdot}$

"Cell (2,1)"

$\bar{y}_{\cdot1\cdot}$ $\bar{y}_{\cdot2\cdot}$ $\bar{y}_{\cdot3\cdot}$

$\bar{y}_{\cdot\cdot\cdot} = 164.833$

Data Decomposition

observations

155	198	235
173	177	229
137	129	155
160	98	132

=

Cell Means

164	187.5	232
"	"	"
148.5	113.5	143.5
"	"	"

\+

Residual

-9	10.5	3.0
9	-10.5	-3.0
-11.5	15.5	11.5
11.5	-15.5	-11.5

6.9

Observations

155	198	235
173	177	229
137	129	155
160	98	132

=

Grand Mean

164.833	"	"
"	"	"
"	"	"
"	"	"

+

Filler

29.667	"	"
"	"	"
-29.667	"	"
"	"	"

+

Proportion

-8.583	-14.333	22.917
"	"	"
"	"	"
"	"	"

+

Filler * Prop

-21.917	7.333	14.583
"	"	"
21.917	-7.333	-14.583
"	"	"

+

Residual

-9	10.5	3.0
9	-10.5	-3.0
-11.5	15.5	11.5
11.5	-15.5	-11.5

Sum of Squares & ANOVA Table

$$\text{Total SS} = 152^2 + 173^2 + \ldots + 132^2 = 344216$$

$$\text{SS}_{\text{mean}} = 12 \times 164.833^2 = 326040.3$$

$$\text{Corrected Total SS} = 18175.67$$

$$\text{SS}_{\text{Filler}} = 29.667^2 + 29.667^2 + \ldots + (-29.667)^2 = 10561.3$$

$$\text{SS}_{\text{Prop}} = (-8.583)^2 + (-8.583)^2 + \ldots + 22.917^2 = 3217.2$$

$$\text{SS}_{\text{Interaction}} = (-21.917)^2 + (-21.917)^2 + \ldots + (-14.583)^2 = 2987.2$$

$$\text{SS}_{\text{Residual}} = (-9)^2 + 9^2 + \ldots + (-11.5)^2 = 1410.0$$

```
Analysis of Variance Table
             Df   Sum Sq Mean Sq F value    Pr(>F)
Mean          1 326040.3
prop          2   3217.2  1608.6  6.8450 0.0282951 *
filler        1  10561.3 10561.3 44.9418 0.0005349 ***
prop:filler   2   2987.2  1493.6  6.3557 0.0329715 *
Residuals     6   1410.0   235.0
Corr. Tot    11 18175.67
Total        12 344216
```

EX SAS & R

- Note that if we treated this as a one-way ANOVA with 6 treatments (instead of a two-way "$2 \cdot 3$ factorial"), then

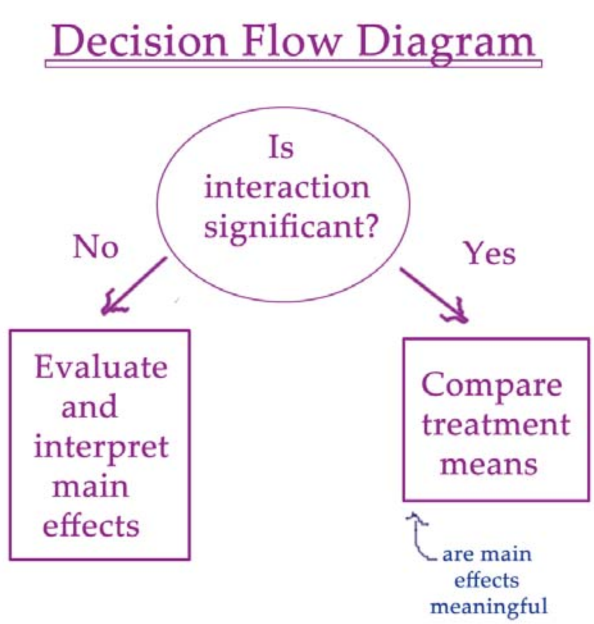
	Df	Sum Sq	Mean Sq	F value	Pr(>F)	
Mean	1	326040.3				
prop	2	3217.2	1608.6	6.8450	0.0282951	*
filler	1	10561.3	10561.3	44.9418	0.0005349	***
prop:filler	2	2987.2	1493.6	6.3557	0.0329715	*
Residuals	6	1410.0	235.0			
Corr. Tot	11	18175.67				
Total	12	344216				

becomes...

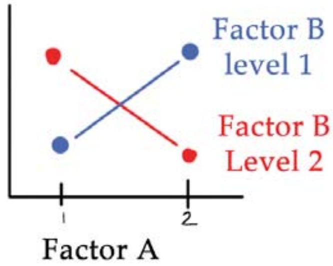
	Df	Sum Sq	Mean Sq	F value	Pr(>F)
Mean	___	________			
Treatment	___	________	______	_______	______
Residuals	___	________	______		
Corr. Tot	___	________			
Total	___	________			

Interpreting Interactions and Main Effects

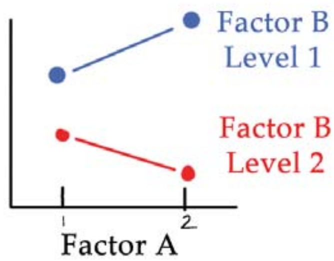
The meaning of a main effect changes if there is a significant interaction.



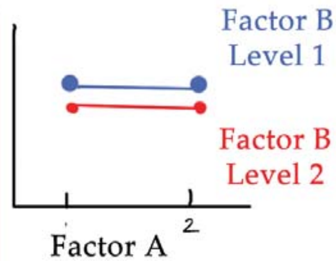
Does the p-value for Factor B properly quantify its importance?



- p-value for interaction is 0.01
- p-value for factor B is 0.90



- p-value for interaction is0.01
- p-value for factor B is 0.01



- p-value for interaction is 0.90
- p-value for factor B is 0.20

What if the # of observations in each cell is just 1?

e.g.:

Proportion of filler

Filler type		25% (1)	50% (2)	75% (3)	
	1	164	187.5	232	194.5 ← $\bar{y}_{1\cdot\cdot}$
	2	148.5	113.5	143.5	135.167 ← $\bar{y}_{2\cdot\cdot}$
		156.25	150.5	187.75	

$\bar{y}_{13\cdot}$ $\quad$ $\bar{y}_{21\cdot}$ $\quad$ $\bar{y}_{\cdot1\cdot}$ $\quad$ $\bar{y}_{\cdot2\cdot}$ $\quad$ $\bar{y}_{\cdot3\cdot}$

$\bar{y}_{\cdot\cdot\cdot} = 164.833$

```
             Df Sum Sq Mean Sq F value Pr(>F)
prop          2 1608.6   804.3
filler        1 5280.7  5280.7
prop:filler   2 1493.6   746.8
Residuals     0    0.0
Warning message:
In anova.lm(wear.lm) :
  ANOVA F-tests on an essentially perfect fit are unreliable
```

Why Replication is So Important

First, let's compare parameters vs. estimates for common models:

- BF[1]: $y_{ij} = \mu + \alpha_i + \varepsilon_{ij}$

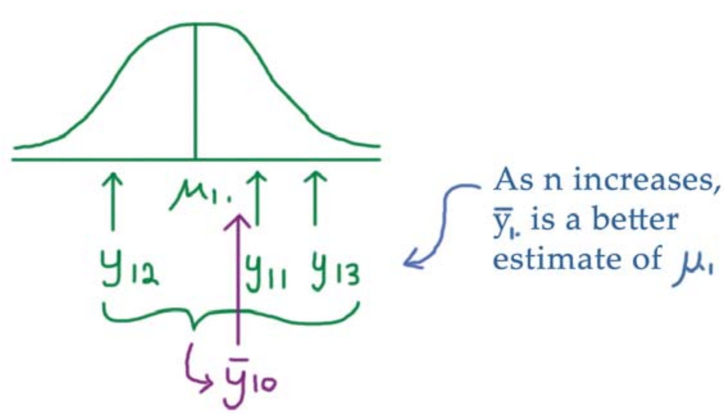
Parameters	Estimates
μ	$\bar{y}_{..}$
$\alpha_i \ (i = 1, \ldots, I)$	$\widehat{\alpha}_i = \bar{y}_{i.} - \bar{y}_{..}$

- BF[2]: $y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \varepsilon_{ijk}$

Parameters	Estimates
μ	$\widehat{\mu} = \bar{y}_{...}$
$\alpha_i \ (i = 1, ..., I)$	$\widehat{\alpha}_i = \bar{y}_{i..} - \bar{y}_{...}$
$\beta_j \ (j = 1, ..., J)$	$\widehat{\beta}_j = \bar{y}_{.j.} - \bar{y}_{...}$
$(\alpha\beta)_{ij} \ (i = 1, ..., I, \ j = 1, ..., J)$	$\widehat{(\alpha\beta)}_{ij} = \bar{y}_{ij.} - (\widehat{\mu} + \widehat{\alpha}_i + \widehat{\beta}_j)$ $= \bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{...}$

Replication is important because:

1. Replication gives more precision to our estimates of model parameters
   - E.g. in a one-factor study, the mean of possible values for units in the first group is $\mu_{1.}$, but we only observe $y_{11}, ..., y_{1n}$



2. Replication gives us information about the errors ε_{ij}, which then allows us to make inference about model parameters.

What can we do if there is not replication?

- Hope that the prop×filler interaction is non-existant/non-significant
- Remember:

$$\text{F-ratio} = \frac{MS_{Interaction}}{MS_{Error}} = \frac{\text{One estimate of chance error variability} + \text{Estimate of variability due to interaction}}{\text{Another estimate of chance error variability}}$$

  - So, if there is no interaction effect...
    - ... $\text{MS}_{\text{interax}}$ is just an estimate of σ^2 (i.e., $\text{MS}_{\text{interax}} \cong \sigma^2$)
    - ... just run the model with no interaction term

Model with interaction:

```
              Df Sum Sq Mean Sq F value Pr(>F)
prop           2 1608.6   804.3
filler         1 5280.7  5280.7
prop:filler    2 1493.6   746.8
Residuals      0    0.0
Warning message:
In anova.lm(wear.lm) :
  ANOVA F-tests on an essentially perfect fit are unreliable
```

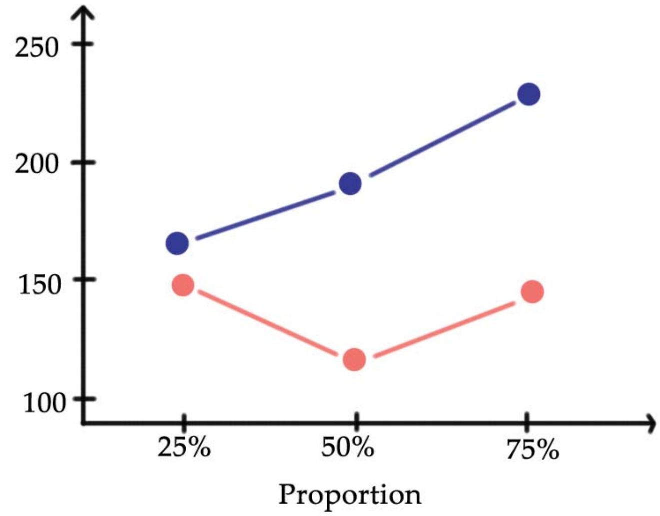
Model without interaction:

```
          Df Sum Sq Mean Sq F value Pr(>F)
prop       2 1608.6   804.3  1.0770 0.4815
filler     1 5280.7  5280.7  7.0711 0.1171
Residuals  2 1493.6   746.8
```

- But how do we know if it's safe to drop the interaction from the model?

Filler type	25%(1)	50%(2)	75%(3)
1	164	187.5	232
2	148.5	113.5	143.5

Proportion of filler



- Should we be dropping the interaction (treating it as error)?

Unbalanced Data

Even though many of our ANOVA formulas require equal n for each treatment combination in a design, we can still do our analyses — we'll just rely on software.

BF[2] (aka 2-way ANOVA):

- Let $n_{11}, n_{12}, \ldots, n_{IJ}$ be the sample sizes in each group with

$$N = \sum_{i=1}^{I} \sum_{j=1}^{J} n_{ij}$$

- $\mathrm{df}_{\text{mean}} = 1;\quad \mathrm{df}_A = I - 1;\quad \mathrm{df}_{\mathrm{B}} = \mathrm{J} - 1;$
  $\mathrm{df}_{AB} = (I-1)(J-1);\quad \mathrm{df}_{\text{error}} = N - IJ$

$\boxed{\text{EX}}$ Cancer data in SAS and R

Things to remember with unbalanced data

1. Standard formula for estimating model parameters are often invalid
2. Order of the factors in the model affects SS, MS, and F-tests

E.g.,

Gender		Type: Blader	Bronchus	Colon	Kidney	Rectum	Stomach
	M						
	F						

Should it matter if which line of SAS code we use below?

```
model logdays = type gender type*gender;
```

OR

```
model logdays = gender type type*gender;
```


It <u>does</u> matter.

- Type I SS in SAS (and SS in R) gives first term in the model the chance to “explain” as much of the variability as possible, with 2nd, 3rd, and later terms a chance to explain what is leftover (Type I SS aka “Sequential SS")
- Sometimes useful to use Type III SS (in SAS), which treats each term as if it were last in the model. That is, what is <u>unique</u> about this term after explaining all others (aka “SS Last”)
  - For Type III, $SS_{\text{Model}} \neq SS_{\text{Gender}} + SS_{\text{Type}} + SS_{\text{Interaction}}$